Projective Geometry

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Abstract

In this talk we discuss the nature, construction, and transformations of projective spaces. We will also discuss an important duality between points and lines, and conclude by showing that several classical geometries are embedded as subgeometries of the projective plane.

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1 What Projective Space Is

Imagine that a camera sitting at the origin of $\mathbb{R}^3$ looks out onto a crowd. The people at the front of the crowd obscure the camera's view of those at the back, and the resulting photo shows only the *projection* of $\mathbb{R}^3$ onto the origin along the camera's lines of sight. In other words, each line of sight becomes a point in the plane of the photograph, and each plane through the origin manifests as a line on the screen. The plane of the picture is a model of the **projective plane** $\mathbb{RP}^2$, which we define as the geometry (in the sense of Klein) whose points are (Euclidean) lines in $\mathbb{R}^3$ passing through the origin, and whose lines are (Euclidean) planes in $\mathbb{R}^3$ through the origin.

One shortfall of physical cameras is that they project rays instead of lines. A physical camera cannot see behind itself, but a *mathematical* camera can, and in the view of such a camera, a crowd behind the origin would also be obscured "behind" the people in the first row: the entire line of sight of the camera would be collapsed to a single point in the photograph. So instead of a whole sphere's worth of viewing angles, $\mathbb{RP}^2$ offers us half of that: every point on the sphere is identified with its antipode. Mathematically, we say that $\mathbb{RP}^2$ consists of the set of 1-dimensional subspaces of $\mathbb{R}^3$, or that it is $\mathbb{R}^3$ mod lines: $\mathbb{RP}^2 = \mathbb{R}^3/\{\ell\} \simeq S^2/\{\pm 1\}$, where ℓ is any (Euclidean) line through the origin.

The notation $\mathbb{RP}^2$ suggests that other spaces $\mathbb{RP}^n$ also exist. Let us briefly discuss $\mathbb{RP}^1$, the set of lines in $\mathbb{R}^2$. This space is constructed by considering two points in $\mathbb{R}^2$ projectively equivalent if they lie on the same line; metaphysically, $\mathbb{RP}^1$ is the Euclidean plane lost at sea, with no notion of distance, caring only about direction. Since we care not for distances, we need a consistent way to normalize the distance data of any point in the plane. One obvious way is to divide the point's coordinates by its distance from the origin; this obtains for us a circle, but is too naive because it fails to identify p with $-p$. A better scheme is to consider any line ℓ not passing through a given point O (the origin). Then the line joining O and p has a unique intersection with ℓ; this intersection is the projective point representing the line Op. Almost all lines are accounted for in this way; the only troublemaker is the unique line through O parallel to ℓ. This line represents the projective point at infinity, which we imagine as infinitely far along ℓ in *both* directions, joining it together and compactifying it into a circle. Hence $\mathbb{RP}^1 \simeq S^1$.

This philosophy extends to higher dimensions: choose a codimension-1 hypersurface $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ not passing through O and "projectivize" the lines through O to projective points by considering their intersections with $\mathbb{R}^{n-1}$, and then deal with what happens at infinity by viewing the parallel hypersurface through O as a copy of $\mathbb{RP}^{n-1}$. This yields an inductive construction of $\mathbb{RP}^n$ which we will formalize in the next section.

2 Coordinates and Transformations

2.1 Projective Coordinates

In the previous section, our construction of $\mathbb{RP}^1$ was obtusely Euclidean in that it made no use of coordinates. Let us now introduce the standard Cartesian coordinates (x, y, z) on $\mathbb{R}^3$ and construct $\mathbb{RP}^2$ by explicitly giving the **projective coordinates** of any $p = (x, y, z) \in \mathbb{R}^3$. As directed above, consider the plane $z = 1$ floating above the origin $O = (0, 0, 0)$; we see that p intersects this plane at coordinates $[p] = [\frac{x}{z} : \frac{y}{z} : 1] \in \mathbb{RP}^2$, where we have used square brackets for added fanciness and colons to hint at the fact that projective coordinates are ratios of the original coordinates. As in the construction of $\mathbb{RP}^1$ above, this scheme uniquely projects almost all lines onto the plane $z = 1$. However, the lines in the plane of the origin are problematic: they never intersect the plane $z = 1$, and moreover they have $z = 0$ which makes their projective coordinates go to infinity. Evidently we supplant the ordinary plane $z = 1$ with more points to make it behave projectively.

What should we add at infinity? We can deduce what shape to add by considering the form that must be taken by the projective coordinates of a point p on the plane $z = 0$. If we give $p = (x, y, 0)$ the coordinates $[p] = [x : y : 0]$, then we do not identify lines through the origin as desired. To fix this, we simply projectivize the whole plane $z = 0$, i.e. fix projective coordinates $[p] = [\frac{x}{y} : 1 : 0]$ on the plane, except for those points with $y = 0$ which have coordinates $[p] = [x : 0 : 0]$. We see that we need to supply the plane $z = 1$ with an entire copy of $\mathbb{RP}^1$, i.e. a circle, at infinity. This makes sense geometrically: a projective plane should just be a plane fenced in by a circle, just as the projective line is in some sense a line fenced in by a point at infinity.

In higher dimensions, the procedure is the same. Start with a huge space $\mathbb{R}^{n+1}$ where points have coordinates $(x_1, ..., x_{n+1})$. Consider the hyperplane $x_{n+1} = 1$, and intersect lines through the origin with this hyperplane to get projective coordinates $[\frac{x_1}{x_{n+1}} : ... : 1]$. The hyperplane $x_{n+1} = 0$ below consists of parallel lines which must be shoved in at infinity. Our coordinate reasoning above suggests that we should view this hyperplane as a lower-dimensional projective space. We therefore get a neat inductive construction:

$$\mathbb{RP}^n \cong \mathbb{R}^{n+1}/\{\ell\} \cong \mathbb{R}^n \sqcup \mathbb{RP}^{n-1} \implies \mathbb{RP}^n = \bigsqcup_{k=0}^{n} \mathbb{R}^k. \tag{2.1}$$

(**N.B.** For pedagogical ease, we have ignored an important technicality: we must remove the origin itself from the definition of the projective space. Its projective coordinates are undefined, and moreoever having an origin contradicts the "lost at sea" philosophy of the projective world.)

2.2 Projective Transformations

Recall that a geometry in the sense of Klein consists of a set and a group of transformations acting on the set. We know that for Euclidean n-space the "full" transformation group is the set $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices (plus translations by arbitrary vectors, if we are working in $\mathbb{R}^n$). What is the corresponding group of transformation matrices for $\mathbb{RP}^n$?

We already know the answer: projective transformations are the same as their Euclidean counterparts, except that we should mod out by all of the matrices that act trivially on projective space. Now two points p_1, p_2 are projectively equivalent if their coordinates satisfy $p_1 = \lambda p_2$ (i.e. they lie on the same line), so any scalar multiple of the identity necessarily acts trivially on projective space. Moreover, this is the only type of matrix that acts trivially because any other would not preserve the equivalence of points above.

We can formalize this construction by declaring two elements $A, B \in GL(n, \mathbb{R})$ equivalent, $A \sim B$ if and only if $A = \lambda B$. Then the set $\text{Pr}(n) = GL(n+1, \mathbb{R})/\sim$ of equivalence classes of matrices is the proper transformation group of projective space. (It is easy to check that this set actually forms a group.) This group is sometimes called the **projective linear group** $PGL(n, \mathbb{R})$, and is often defined by $PGL(n, \mathbb{R}) = GL(n, \mathbb{R})/Z(GL(n, \mathbb{R}))$, the general linear group modulo its center. The center of a group consists of all elements that commute with everything in the group, and it is a cute linear algebra exercise to show that $Z(GL(n, \mathbb{R}))$ consists only of scalar multiples of the identity.

3 Projective Duality

In the Euclidean geometry of 3-dimensional space and multivariable calculus, many students learn that every line uniquely determines a plane orthogonal to that line; likewise, every plane determines a normal direction yielding this line. In projective geometry, this duality between lines and planes in $\mathbb{R}^3$ is upgraded: every line becomes a point and every plane a line, so points and lines are naturally dual to each other. In some sense, the question "is there a 2-dimensional geometry where points and lines are naturally dual?" is answered directly by the projective plane.

We could have predicted hints of this duality even in the Euclidean plane. For instance, if we view the plane as consisting of a set of points, then a line is a particular subset of those points; likewise, if we view the plane as consisting of the set of all lines, a point is just a particular subset of those lines: namely, those that all intersect in a given point. Trying to formulate "dual axioms" of Euclidean geometry by interchanging the roles of points and lines, however, proves difficult.

To make the case for projective geometry, let us introduce the notion of incidence. We say two lines ℓ, m are **incident** at point P if they intersect at P, and points P and Q are **incident** at line ℓ if ℓ passes through P and Q. In this language, we can formulate two fundamental statements in projective geometry:

1. One and only one line is incident to two distinct points;

2. One and only one point is incident to two distinct lines.

We see that exchanging "point" and "line" also exchanges statements 1 and 2, so a projective geometry where the meanings of point and line are exchanged is no different than the geometry we have considered.

To formalize this claim, define the **dual geometry** to $(\mathbb{RP}^2 : \mathrm{Pr}(2))$, denoted $(\overline{\mathbb{RP}^2} : \mathrm{Pr}(2))$, to be the geometry whose points are planes of $\mathbb{R}^3$ through the origin, and where the "intersection" of two points (i.e. Euclidean planes) will be called the (dual-projective) line passing through the points, which turns out to be a Euclidean line as well. We then have a bombshell result:

Theorem 3.1 (Projective duality)**.** *The geometries* $(\mathbb{RP}^2 : Pr(2))$ *and* $(\overline{\mathbb{RP}^2} : Pr(2))$ *are isomorphic: there is a bijection* $D\colon \mathbb{RP}^2 \to \overline{\mathbb{RP}^2}$ *compatible with the action of* $Pr(2)$*.*

Proof. In the spirit of Euclidean duality, note that the point $[a : b : c] \simeq [\frac{a}{c} : \frac{b}{c} : 1]$ in $\mathbb{RP}^2$ naturally corresponds to (i.e. uniquely identifies) the dual "point" given by the Euclidean plane $ax + by + cz = 0$. Then construct D by sending $[a : b : c] \mapsto \{ax + by + cz = 0\}$. This map is obviously bijective; it remains to show compatibility under the action of Pr(2). But this is obvious: if some $g \in \mathrm{Pr}(2)$ sends $[a : b : c] \mapsto [a' : b' : c']$, then in the dual geometry $\overline{g}$ will send $\{ax + by + cz = 0\} \mapsto \{a'x + b'y + c'z = 0\}$. Hence there is a commutative square with D along the horizontals and g along the verticals proving compatibility. □

Observe that the map D is an *involution*: applying D twice gets us back where we came from, i.e. $D^2 = \mathrm{id}_{\mathbb{RP}^2}$. Moreover, it preserves the notion of incidence: if two points A, B are incident to a line ℓ in $\mathbb{RP}^2$, then their images under D are lines $D(A),\ D(B)$ incident to the point $D(\ell)$ in $\overline{\mathbb{RP}^2}$. Therefore:

Corollary 3.2. *There is a bijection between the set of lines and the set of points of* $\mathbb{RP}^2$ *that preserves incidence and takes any theorem of* $\mathbb{RP}^2$ *geometry to a theorem of* $\mathbb{RP}^2$*.*

4 "Projective Geometry is All Geometry"

We now present a wonderful and perhaps surprising fact about $\mathbb{RP}^2$: it "contains" the three other continuous geometries we've studied. To formalize the notion of containment, recall our definition of equivalent geometries: two geometries $(X : G)$ and $(Y : H)$ are **equivalent** or **isomorphic** if (a) there is a bijection $f\colon X \to Y$, (b) there is an isomorphism of groups $\phi\colon G \to H$, and (c) the bijection f is **equivariant**, i.e. it respects the group actions on both geometries: for any $g \in G$ and $x \in X$, $f(g \cdot x) = \phi(g) \cdot f(x)$. We can weaken this definition to that of a **subgeometry**: now in addition to compatibility with the group action, we merely require that f and ϕ be monomorphisms (i.e. injective).

We endeavor to sketch the proof of the following theorem:

Theorem 4.1. *The following geometries are subgeometries of* $(\mathbb{RP}^2 : Pr(2))$*:*

1. *the Euclidean plane* $(\mathbb{R}^2 : Iso(2))$*;*
2. *the hyperbolic plane* $(\mathbb{H}^2 : \mathbb{R}M\ddot{o}b)$*;*
3. *the elliptic plane* $(S^2/\{\pm 1\} : O(3))$*.*

Before we prove the theorem, we make a few remarks:

- Recall that we can view $\mathbb{RP}^2$ variously as the plane $z = 1$ in $\mathbb{R}^3$ supplied with a circle (i.e. $\mathbb{RP}^1$) at infinity, or as a sphere of lines of sight with antipodal points identified.
- We will see that in the first view, the Euclidean plane simply comes from the "non-infinity" part of $\mathbb{RP}^2$. Similarly, we will situate the Cayley-Klein model of hyperbolic geometry inside the plane $z = 1$ by considering the unit disk in that plane.
- The elliptic plane has the same structure as $\mathbb{RP}^2$ in the second view; nevertheless, we can also place the Riemann sphere on top of the plane $z = 1$ in the first view to show that the elliptic plane fits neatly into projective space.
- The main difficulty proceeding forward will be to show that the transformation groups of each model actually embed into Pr(2), and that their actions are compatible with the inclusions we construct.

4.1 The Euclidean Plane

Let Π denote the plane $z = 1$ in $\mathbb{R}^3$, and let Λ_∞ denote the circle at infinity surrounding it. Then as sets $\mathbb{RP}^2 = \Pi \cup \Lambda_\infty$, and $\mathbb{R}^2 = \Pi \subset \mathbb{RP}^2$. Therefore define the inclusion $f \colon \mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ in the "obvious" way by $f(x_1, x_2) = [x_1 : x_2 : 1]$.

We will construct a map $\tilde{\phi} \colon \mathrm{Iso}(2) \to \mathrm{Pr}(2)$ in the following way. Start with an element $g \in \mathrm{Iso}(2)$ that maps some point $p \in \Pi$ to $g(p) \in \Pi$. Then let $\phi(g)$ be the projective transformation that drags the entire line connecting the origin O to p to the line connecting O to $g(p)$; in symbols, $g \colon p \mapsto g(p) \implies \phi(g) \colon \overline{Op} \mapsto \overline{Og(p)}$. It is obvious that f is injective, and from the discussion of homogeneous coordinates above we can see that ϕ is also injective. (Here Λ_∞ remains untouched, so the correspondence is between lines passing through $z = 1$ exactly once, and their points of intersection.) It is also easy to see that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ and that the Iso(2)-action is respected by the inclusion f.

4.2 The Hyperbolic Plane

Once again, take our model of projective geometry to be $\mathbb{RP}^2 = \Pi \cup \Lambda_\infty$. Recall that the hyperbolic plane ($\mathbb{H}^2$: $\mathbb{R}$Möb) can be realized as the Cayley-Klein model ($D^2 : M$), where D^2 is the open unit disk and M is the group of isometries with metric given by $d(A, B) = \frac{1}{2}|\log(\langle A, B, X, Y\rangle)|$. Situate $D \subset \Pi \subset \mathbb{R}^3$ as the open unit disk centered at the point directly above the origin O on the plane $z = 1$, and define the map $f \colon D^2 \hookrightarrow \mathbb{RP}^2$ by inclusion: $f(x_1, x_2) = [x_1 : x_2 : 1]$.

To construct the monomorphism $\phi \colon M \to \mathrm{Pr}(2)$, start with an isometry $g \in M$ and four points $A, B, C, D \in D^2$ in general position (i.e. no three of them collinear). Under g, they are sent to four other points $gA, gB, gC, gD \in D^2$. Denote their inclusions in $\mathbb{RP}^2$ by

$$\begin{aligned} &A_1 = f(A), \quad &&B_1 = f(B), \quad &&C_1 = f(C), \quad &&D_1 = f(D); \\ &A_2 = f(gA), \quad &&B_2 = f(gB), \quad &&C_2 = f(gC), \quad &&D_2 = f(gD). \end{aligned} \tag{4.1}$$

To define the transformation $\phi(g) \in \mathrm{Pr}(2)$, recall that a unique projective transformation $\tilde{g} \in \mathrm{Pr}(2)$ takes four points in general position to four points again in general position. Since g itself preserves general position, we may apply this fact to define $\phi(g) = \tilde{g}$.

It is clear that f is injective, and by construction so is ϕ. We actually still need to verify that $\phi(g)$ restricts to $f(D^2) \subset \mathbb{RP}^2$ in a way that coincides with g. We omit the details, but we will mention that this follows from the fact that projective transformations preserve the cross-ratio of any four collinear points, and therefore preserve the distance d between points in $f(D^2)$. But since g is an isometry, it must coincide with the restriction of $\phi(g)$ to non-collinear points in $f(D^2)$, so it coincides globally. This proves the desired compatibility.

4.3 The Elliptic Plane

We mentioned above that both $\mathbb{RP}^2$ and the elliptic plane look like $S^2/\{\pm 1\}$ as sets, so it is clear that they are in bijection. However, we will still use the model $\mathbb{RP}^2 = \Pi \cup \Lambda_\infty$ and place a copy of the sphere S^2 directly above it, so that the south pole touches the point directly above O on Π. The map $f\colon S^2/\{\pm 1\} \hookrightarrow \mathbb{RP}^2$ will be constructed by extending a diameter through the sphere until it intersects Π; in this way every point and its antipode on S^2 are projected onto a unique point on $\Pi \cup \Lambda_\infty$. In fact, lines on the elliptic plane—great circles on the sphere—are projected to rectilinear lines on the projective plane, and the equator parallel to $z = 1$ is mapped onto the projective line at infinity.

The construction of $\phi\colon O(3) \to \mathrm{Pr}(2)$ is somewhat involved, so we will omit the details. In brief, however, it is almost identical to the construction for the hyperbolic plane. We start by constructing four points $A, B, C, D \in S^2/\{\pm 1\}$ and consider their images A', B', C', D' under some transformation $g \in O(3)$. After projecting these points and their images onto $\mathbb{RP}^2$ and ensuring that they land in general position, we see again that there exists a unique projective transformation $\phi(g)$ taking one set of projections to the other.

We leave as an exercise to show that this construction forces ϕ to be injective and compatible with group actions, so that the elliptic plane is indeed a subgeometry of $\mathbb{RP}^2$.